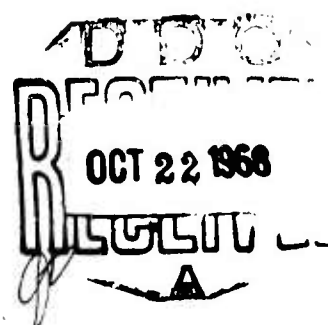


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ON THE EXPLICIT SOLUTION
OF A SPECIAL CLASS OF
LINEAR ECONOMIC MODELS

by

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Introduction

In this paper we give explicit solutions for a class of linear (inequality) economic models heretofore treated by linear programming. These solutions are quite elementary and the decision criteria given are easily interpreted. Integrality constraints are naturally incorporated in our solutions, without resorting to special integer programming algorithms. The explicit solutions may involve the inversion of a single matrix but even this is not necessary for the three examples in capital budgeting, production planning and input-output analysis given below.

§0: Notations

The following notations are used:

R^n the n -dimensional real vector space;

I_n the $n \times n$ identity matrix;

For any $x, y \in R^n$:

x_i denotes the i^{th} component of x ;

$x \geq y$ denotes $x_i \geq y_i, i = 1, \dots, n$;

For any $m \times n$ matrix A , denote by:

a_{ij} the element in row i , column j ;

A^t the transpose of A ;

A^{-1} the inverse of A ;

§1. Interval linear programming

An interval linear program, abbreviated IP, is a problem of the form:

$$(1) \quad \text{maximize } c^t u$$

s.t.

$$(2) \quad a \leq Au \leq b$$

where the vectors c , a , b and the matrix A are given. Any IP can be written as an ordinary linear program and conversely any bounded linear program is equivalent to an IP; thus interval programming is an alternate formulation of linear programming. For a survey of IP and selected applications see [3] and [5].

If A is of full row rank then IP can be solved explicitly using a generalized inverse of A , [2]. The general case was solved iteratively in [5], [6] and [7]. Conditions for the explicit solution of the general IP were given in [9]. In the applications considered here the matrix A is nonsingular so that ordinary, rather than generalized, inverses may be used. The main theorem of [2] reduces in this case to:

Theorem: Let the IP (1), (2) be given with A $n \times n$ nonsingular and

$$(3) \quad a \leq b$$

Then the optimal solutions of (1), (2) are all the vectors of the form

$$(4) \quad u^* = A^{-1} \bar{u}^*$$

where the components of $\bar{u}^* \in R^n$ are defined by

$$(5) \quad \bar{u}_j^* = \begin{cases} b_j \\ \theta_j a_j + (1-\theta_j) b_j \\ a_j \end{cases} \quad \text{if } (c^t A^{-1})_j \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

for $j = 1, \dots, n$ and $0 \leq \theta_j \leq 1$
for those j with $(c^t A^{-1})_j = 0$.

Proof: Substituting

$$(6) \quad \bar{u} = Au$$

in (1), (2) we obtain the equivalent problem

$$(7) \quad \text{maximize } c^t A^{-1} \bar{u}$$

s.t.

$$(8) \quad a \leq \bar{u} \leq b$$

whose optimal solutions are the vectors \bar{u}^* given by (5). The reverse substitution gives (4). QED

§2: A class of linear economic models explicitly solvable by interval programming.

The problems considered here are of the form

$$(9) \quad \text{maximize } g^t x + h^t y$$

s.t.

$$(10) \quad p \leq x \leq q$$

$$(11) \quad r \leq Bx + Cy \leq s$$

where the vectors $g, p, q \in R^n$; $h, r, s \in R^m$ and the $m \times n$ matrix B and the $m \times m$ matrix C are given. C is assumed nonsingular.

This problem is of the form (1), (2) with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad A = \begin{pmatrix} I_n & : & 0 \\ \dots & & \dots \\ B & : & C \end{pmatrix}, \quad a = \begin{pmatrix} p \\ r \end{pmatrix}, \quad b = \begin{pmatrix} q \\ s \end{pmatrix}$$

and $c = \begin{pmatrix} g \\ h \end{pmatrix}$. From the (assumed) nonsingularity of C it follows that A is nonsingular, and indeed

$$(12) \quad A^{-1} = \begin{pmatrix} I_n & : & 0 \\ \dots & & \dots \\ -C^{-1}B & : & C^{-1} \end{pmatrix}$$

Reasoning as in the proof of the theorem in § 1 we substitute

$$(13) \quad \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$$

in (9) - (11) and use (12) to obtain the equivalent problem

$$(14) \quad \text{Maximize } (g^t - h^t C^{-1} B) \bar{x} + h^t C^{-1} \bar{y}$$

s.t.

$$(15) \quad p \leq \bar{x} \leq q$$

$$(16) \quad r \leq \bar{y} \leq s.$$

The optimal solutions of (14) - (16) are vectors of the form $\begin{pmatrix} \bar{x}^* \\ \bar{y}^* \end{pmatrix}$ given by

$$(17) \quad \bar{x}_i^* = \begin{cases} q_i \\ \theta_i p_i + (1 - \theta_i) q_i \\ p_i \end{cases} \quad \text{if } g_i - (h^t C^{-1} B)_i \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

and

$$(18) \quad \bar{y}_j^* = \begin{cases} s_j \\ \theta_j r_j + (1 - \theta_j) s_j \\ r_j \end{cases} \quad \text{if } (h^t C^{-1})_j \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

($i = 1, \dots, n$; $j = 1, \dots, m$) where $0 \leq \theta_i \leq 1$, $0 \leq \theta_j \leq 1$.

Using (17), (18) and the reverse substitution of (13), the optimal solutions of (9)-(11) may thus be written explicitly as

$$(19) \quad x^* = \bar{x}^*$$

$$(20) \quad y^* = -C^{-1} B \bar{x}^* + C^{-1} \bar{y}^* .$$

§ 3: A Capital budgeting model

Consider the problem faced by a firm which must select investment projects from the set of all possible projects when the yearly cash flow associated with each project is known. Assume that a perfect capital market exists so that a firm may borrow or lend any amount of money at a given interest rate. Let

M = the number of time intervals in the planning horizon;

N = the number of potential projects;

s_j = the amount of money available for investment in period j which is generated by activities not included in the model;

b_{ji} = the flow of money associated with project i during period j
 ($b_{ji} > 0$ denotes expenditure or outflow, $b_{ji} < 0$ denotes inflow);

y_j = the amount borrowed (< 0) or lent (> 0) in period j ;

c = the interest rate ;

g_i = the horizon value of all cash flows subsequent to the horizon associated with project i ;

x_i = the fraction of investment project i undertaken.

The problem is to maximize the sum of all discounted flows subsequent to the horizon plus the difference between lending and borrowing at the horizon, subject to the condition that outflows of money in each period must not be greater than inflows. That is

$$(21) \quad \text{maximize} \quad \sum_{i=1}^N g_i x_i + y_M$$

s.t.

$$(22) \quad 0 \leq x_i \leq 1, \quad i = 1, \dots, M$$

$$(23) \quad 0 \leq \sum_{i=1}^N b_{1i} x_i + y_1 \leq s_1$$

$$(24) \quad 0 \leq \sum_{i=1}^N b_{ji} x_i - (1+\zeta) y_{j-1} + y_j \leq s_j, \quad j = 2, \dots, M.$$

This problem has the form of (9) - (11) with $r = p = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$,

$$q = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \quad h = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ -(1+\zeta) & 1 & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & \dots & \dots & 0 \\ 0 & \dots & \dots & 0 & -(1+\zeta) & 1 \end{pmatrix}, \quad \text{and} \quad C^{-1} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ (1+\zeta) & 1 & \dots & \dots & \vdots \\ (1+\zeta)^2 & (1+\zeta) & 1 & \dots & \vdots \\ \vdots & \dots & \dots & \dots & \vdots \\ (1+\zeta)^{M-1} & \dots & \dots & (1+\zeta) & 1 \end{pmatrix}$$

From (19), the optimal solution or decision rule for accepting projects can be written explicitly as follows:

$$(25) \quad x_i^* = \begin{cases} 1 \\ \theta_i \\ 0 \end{cases} \quad \text{if} \quad g_i - \sum_{j=1}^{M-1} (1+\zeta)^j b_{(M-j)i} - b_{Mi} \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

for $i = 1, \dots, N$ where $0 \leq \theta_i \leq 1$ is arbitrary.

This model is studied in Weingartner [8, p. 141-147] (generalizing the model of Lorie and Savage [4]) where a solution (25) with all $\theta_i = 1$ is derived using duality theory, and where it is suggested on realistic grounds [8, p.142] that the x_i should be restricted to integer values (i.e. prohibit partial projects) and that integer linear programming be used. Our result permits us trivially to satisfy the integrality constraints since θ_i in (25) can be chosen as $\theta_i = 0$ or 1. Also our solution is not affected by the absence of satisfactory duality theory in integer programming.

If one recalls the meanings of b_{ji} , g_i , and ζ , the decision rule for accepting projects has a straight-forward economic interpretation: project i is accepted (i.e. $x_i = 1$) if the total value of all cash flows discounted to the horizon (interest rate = ζ) plus the value of flows subsequent to the horizon is positive. Observe that the explicit solution can be obtained for any linear objective function^{1/} and that the sensitivity of project selection decisions to changes in g_i and b_{ji} are readily available from (25).

The perfect capital market assumption effectively makes the decision for each project independent of the action taken on other projects. Without this assumption or with different interest rates for borrowing and lending, the theory of section 1 does not apply directly because (24) is then changed to give an IP (1) (2) with a singular coefficient matrix A . Interval linear programming techniques exist for this case too, (see [5], [6], and [7]).

^{1/} A related "utility maximization" model proposed by Baumol and Quandt [1] is also covered by the theorem of §1 provided that the variables are permitted to take on negative values (i.e. both consumption and "investment" are permitted in each period).

S4: A production planning model

Suppose that a firm produces several (N) products using several (M) different resources. The firm has available in each decision period a supply of each resource (s_j) from its own sources. In addition, it can purchase additional resources at h_j dollars per unit of resource j on the open market. If the total supply of resource j is not required, the excess can be sold at the price of h_j dollars per unit. The sale of one unit of product i yields a profit of g_i dollars. The firm wants to select the quantity of each product to produce so that profit is maximized. In addition to the above notation let:

b_{ji} = the quantity of resource j required per unit of product i produced;

x_i = the units of product i produced in the decision period;

y_j = the units of resource j to be purchased (< 0) or sold (> 0);

q_i = the maximum quantity of product that can be produced in period i (e.g. due to demand limitations or institutional restrictions).

The production planning model can be written

$$(26) \quad \text{maximize} \quad \sum_{i=1}^N g_i x_i + \sum_{j=1}^M h_j y_j$$

s.t.

$$(27) \quad 0 \leq x_i \leq q_i, \quad i = 1, \dots, N$$

$$(28) \quad 0 \leq \sum_{i=1}^N b_{ji} x_i + y_j \leq s_j, \quad j = 1, \dots, M.$$

This problem has the form of problem

$$(9) - (11) \text{ with } p = r = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \text{ and } C = C^{-1} = I_M.$$

From (19) the optimal amount of product 1 to produce is

$$(29) \quad x_1^* = \begin{cases} q_1 \\ \theta_1 q_1 \\ 0 \end{cases} \text{ if } g_1 = \sum_{j=1}^N h_j b_{j1} \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

with $0 \leq \theta_1 \leq 1$ arbitrary. The optimal amount of resource j can also be written explicitly using (20).

Here again, the market for the limited resources yields a direct valuation of their worth to the firm. The net profit from the production of a unit of product 1 is g_1 minus the total market value of the limited resources employed in its production. If this net profit is positive, x_1 should be set at its upper limit as shown in (29).

S. C. Littlechild observed that (29) can be alternatively obtained in two stages: Since the coefficients of y_j , $j = 1, \dots, M$, in (26) and (28) are positive the optimal y^* must satisfy:

$$\sum_{i=1}^N b_{ji} x_i + y_j^* = \begin{cases} 0 \\ \theta_j s_j \\ s_j \end{cases} \text{ if } h_j \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases} \quad (j = 1, \dots, M)$$

which if substituted, for y_j , in (26) gives (29).

Such a two stage solution is possible for problems (9), (10), (11) with special C , e.g. diagonal nonsingular.

§ 5: An input-output model

The static Leontief input-output model of an economy is often cast in the form of a planning model. This version can be solved explicitly

provided imports and exports are permitted from some perfect market.

That is, we assume that any amount of a product can be imported or exported at a given price. Let

N = the number of products included in the model;

M_i = the number of different methods to produce product i ;

$x_i^{(k)}$ = the amount of product i to be produced by method k ($1 \leq k \leq M_i$);

y_i = the amount of product i to be imported (> 0) or exported (< 0);

$d_{ij}^{(k)}$ = the amount of product i used to produce one unit of product j
by the k^{th} method;

$q_i^{(k)}$ = the capacity for production of product i by method k ;

$r_i = s_i$ = the exogenous demand for product i ;

$g_i^{(k)}$ = "profit" per unit of product i produced by method k ;

h_i = unit "cost" of importing and exporting product i ;

The problem to be solved is:

$$(30) \quad \text{maximize} \quad \sum_{i=1}^N \sum_{k=1}^{M_i} g_i^{(k)} x_i^{(k)} - \sum_{i=1}^N h_i y_i$$

subject to

$$(31) \quad 0 \leq x_i^{(k)} \leq q_i^{(k)}$$

$$i = 1, \dots, N; \quad k = 1, \dots, M_i$$

$$(32) \quad r_i \leq \sum_{k=1}^{M_i} x_i^{(k)} - \sum_{j=1}^N \sum_{k=1}^{M_j} d_{ij}^{(k)} x_j^{(k)} + y_i \leq r_i$$

$$i = 1, \dots, N.$$

In this case let

$$x = \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_1^{(M_1)} \\ \vdots \\ x_N^{(1)} \\ \vdots \\ x_N^{(M_N)} \end{pmatrix}, \quad g_i = \begin{pmatrix} g_1^{(1)} \\ \vdots \\ g_1^{(M_1)} \\ \vdots \\ g_N^{(1)} \\ \vdots \\ g_N^{(M_N)} \end{pmatrix}, \quad q_i = \begin{pmatrix} q_1^{(1)} \\ \vdots \\ q_1^{(M_1)} \\ \vdots \\ q_N^{(1)} \\ \vdots \\ q_N^{(M_N)} \end{pmatrix}$$

so that (30) - (32) has the form of problem (9)-(11) with

$$p = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \quad C = I_N \text{ and}$$

$$B = \begin{pmatrix} (1 - d_{11}^{(1)}) \dots (1 - d_{11}^{(M_1)}) & d_{12}^{(1)} \dots d_{12}^{(M_2)} & \dots & d_{1N}^{(1)} \dots d_{1N}^{(M_N)} \\ d_{21}^{(1)} \dots d_{21}^{(M_2)} & (1 - d_{22}^{(1)}) \dots (1 - d_{22}^{(M_2)}) & \dots & d_{2N}^{(1)} \dots d_{2N}^{(M_N)} \\ \dots & \dots & \dots & \dots \\ d_{N1}^{(1)} \dots d_{N1}^{(M_1)} & d_{N2}^{(1)} \dots d_{N2}^{(M_2)} & \dots & (1 - d_{NN}^{(1)}) \dots (1 - d_{NN}^{(M_N)}) \end{pmatrix}$$

From (5) the optimal production levels are

$$(33) \quad x_i^{(k)} = \begin{cases} q_i^{(k)} \\ \theta_i q_i^{(k)} \\ 0 \end{cases} \quad \text{if} \quad g_i^{(k)} + h_i (1 - d_{ii}^{(k)}) + \sum_{j=1, j \neq i}^N h_j d_{ji}^{(k)} \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases}$$

with $0 \leq \theta_i^{(k)} \leq 1$ arbitrary.

Again the decision rule, (33), represents a "net profit" concept similar to that of the production planning model. Of course, "profit" may be measured in some units other than dollars to reflect the planner's evaluation of the different products.

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